HW 2

PHYS 425/525

Griffiths 2.5

Because the z axis is aligned with the center of the circle, $E_x(0,0,z) = E_y(0,0,z) = 0$ because for each charge on the circle, there is the same charge on the opposite side of the circle whose contribution cancels after integrating to obtain these components. Now for each location z,

$$dE_z(0,0,z) = \frac{\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2+z^2}\lambda r d\phi$$

$$E_z(0,0,z) = \frac{\lambda}{2\varepsilon_0}\frac{zr}{\left(r^2+z^2\right)^{3/2}}$$

$$= \frac{Q_{tot}}{4\pi\varepsilon_0}\frac{z}{\left(r^2+z^2\right)^{3/2}}$$

Notice at large z, the field decreases as $1/r^2$ as it should. The field also vanishes on the plane of the loop.

Griffiths 2.6

Simply integrate the result of the previous problem over r.

$$dE_z(0,0,z) = \frac{\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2+z^2}\sigma r dr d\phi$$

$$E_z(0,0,z) = \int_0^R \frac{\sigma}{2\varepsilon_0}\frac{zrdr}{\left(r^2+z^2\right)^{3/2}} = \frac{\sigma}{4\varepsilon_0}\left[-\frac{2z}{\left(r^2+z^2\right)^{1/2}}\right]\Bigg|_0^R$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\left(R^2+z^2\right)^{1/2}}\right]$$

$$= \frac{Q_{tot}}{4\pi\varepsilon_0}\frac{2}{R^2}\left[1-\frac{z}{\left(R^2+z^2\right)^{1/2}}\right]$$

For $R \gg z$ the field must look like the infinite plane result obtained from Gauss's law. It does! In the other limit

$$\left[1-\frac{z}{\left(R^2+z^2\right)^{1/2}}\right]\to 1-\frac{1}{\left(1+R^2/z^2\right)^{1/2}}\approx 1-\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)=\frac{1}{2}\frac{R^2}{z^2}$$

So

$$E_z\left(0,0,z\right)\to\frac{Q_{tot}}{4\pi\varepsilon_0}\frac{1}{z^2}.$$

Far enough away from the disc, the field reduces to that of a point charge!

Griffiths 2.9

Using the spherical divergence on the front cover of Griffiths

$$\nabla\cdot\boldsymbol{E}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^5\right)=5kr^2$$
$$\rho\left(r\right)=5kr^2\varepsilon_0$$

We can determine the charge inside each radius either by integrating the charge density with the spherical volume element, or by Gauss's law. Integrating using the volume element yields

$$Q_{inside}\left(r\right)=\int_0^r\int_{-1}^1\int_0^{2\pi}\rho\left(r'\right)r'^2dr'd\cos\theta d\varphi$$
$$=4\pi 5k\varepsilon_0\int_0^r r'^4dr'=4\pi k\varepsilon_0 r^5$$

Gauss's Law yields

$$Q_{inside}\left(r\right)=\varepsilon_0\int_{S(r)}\boldsymbol{E}\cdot\hat{n}da=\varepsilon_0 kr^3 4\pi r^2$$

Of course, the results are the same.

Griffiths 2.18

For an infinite slab there is only field in the y -direction. Using Gauss's Law using a cube of size $2y$ centered on the origin of the figure, and the fact that by symmetry $E_y\left(y=-|y|\right)=-E_y\left(y=+|y|\right)$

$$E_y\left(y=|y|\right)(2y)^2-E_y\left(y=-|y|\right)(2y)^2=\frac{\rho}{\varepsilon_0}(2y)^3$$

$$\therefore E_y\left(y=|y|\right)=\frac{\rho}{\varepsilon_0}y \qquad E_y\left(y=-|y|\right)=-\frac{\rho}{\varepsilon_0}|y| \qquad |y|\le d$$

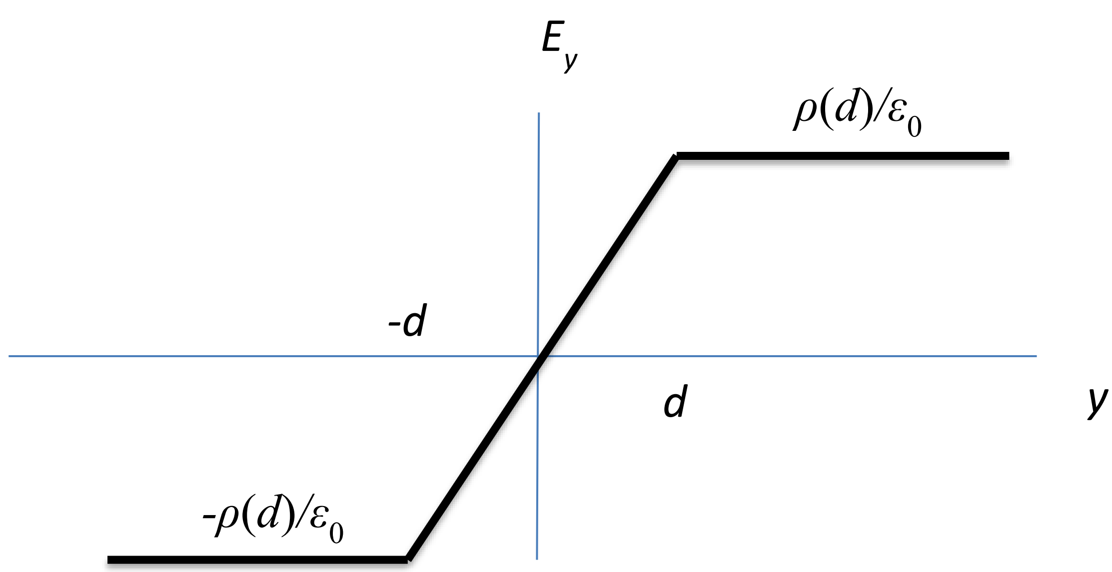
Also

$$E_y\left(y=|y|\right)(2y)^2-E_y\left(y=-|y|\right)(2y)^2=\frac{\rho}{\varepsilon_0}(2y)^2\,2d$$

$$\therefore E_y\left(y=|y|\right)=\frac{\rho}{\varepsilon_0}d \qquad E_y\left(y=-|y|\right)=-\frac{\rho}{\varepsilon_0}d \qquad |y|\ge d$$

The field becomes uniform outside of the slab! The expression for the final answer may be a bit confusing as there is no 2 as in the result for an infinite plane of charge. Remember though that the "effective" surface density is $\sigma_{eff}=\rho 2d$ by the dimensions that Griffiths used to define the problem. Thus $E_y=\sigma_{eff}/2\varepsilon_0$ DOES apply for this problem too!

Here is a plot



The significant things to notice are the field vanishes on the *x-z* plane, as it must by symmetry, it grows linearly with the displacement from the plane, and it saturates to a constant value once $|y|>d$.

Griffiths 2.23

By Gauss's law applied to a cylinder of radius s and length L

$$E_s 2\pi s L = \frac{\lambda L}{\varepsilon_0}$$

$$E_s = \frac{\lambda}{2\pi\varepsilon_0 s}.$$

Defining the zero of the potential is tricky because the field blows up at the origin and does not integrate to a convergent integral at infinity. However, aside from these problem locations, a potential may be defined by choosing a radius s_0 for the zero location and integrating

$$\phi(s) = -\int_{s_0}^{s} E_s(s')\,ds' = -\frac{\lambda}{2\pi\varepsilon_0}\int_{r_0}^{s}\frac{ds'}{s'} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s}{s_0}.$$

Differentiating to get the field

$$E_s = -\frac{\partial\phi}{\partial s} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{(s/s_0)}\frac{1}{s_0} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}.$$

Griffiths 2.22 (needed for 2.35)

Outside the sphere the potential is

$$\frac{Q_{tot}}{4\pi\varepsilon_0}\frac{1}{r}.$$

Inside the sphere edge the field is

$$E_r = \frac{Q_{tot}}{4\pi\varepsilon_0}\frac{r}{R^3}$$

The potential is

$$\phi(r) = \frac{Q_{tot}}{4\pi\varepsilon_0 R} - \int_R^r \frac{Q_{tot} r'}{4\pi\varepsilon_0 R^3}dr' = \frac{Q_{tot}}{4\pi\varepsilon_0 R} - \frac{Q_{tot}\left(r^2 - R^2\right)}{8\pi\varepsilon_0 R^3}$$

$$= \frac{Q_{tot}}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right).$$

Griffiths 2.35

$$\begin{aligned} W &= \frac{1}{2}\int \rho\phi dV \\ &= \frac{1}{2}\frac{3Q_{tot}}{4\pi R^3}\frac{Q_{tot}4\pi}{8\pi\varepsilon_0 R}\int_0^R\left(3-\frac{r^2}{R^2}\right)r'^2dr' \\ &= \frac{1}{2}\frac{Q_{tot}^2}{4\pi\varepsilon_0}\frac{3}{2R^4}\left(R^3-\frac{R^3}{5}\right)=\frac{3}{5}\frac{Q_{tot}^2}{4\pi\varepsilon_0 R} \end{aligned}$$

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int E^2 dV \\ &= \frac{\varepsilon_0}{2}4\pi\left(\frac{Q_{tot}}{4\pi\varepsilon_0}\right)^2\left[\int_0^R\frac{r'^4}{R^6}dr'+\int_R^\infty\frac{r'^2}{r'^4}dr'\right] \\ &= \frac{Q_{tot}^2}{4\pi\varepsilon_0}\frac{1}{2}\left[\frac{R^5}{5R^6}+\frac{1}{R}\right]=\frac{3}{5}\frac{Q_{tot}^2}{4\pi\varepsilon_0 R} \end{aligned}$$

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int E^2 dV+\frac{\varepsilon_0}{2}\oint \phi\mathbf{E}\cdot\hat{n}da \\ &= \frac{\varepsilon_0}{2}4\pi\left(\frac{Q_{tot}}{4\pi\varepsilon_0}\right)^2\left\{\left[\int_0^R\frac{r'^4}{R^6}dr'+\int_R^a\frac{r'^2}{r'^4}dr'\right]+\frac{a^2}{a^3}\right\} \qquad a>R \\ &= \frac{Q_{tot}^2}{4\pi\varepsilon_0}\frac{1}{2}\left[\frac{R^5}{5R^6}+\frac{1}{R}-\frac{1}{a}+\frac{1}{a}\right]=\frac{3}{5}\frac{Q_{tot}^2}{4\pi\varepsilon_0 R} \qquad a>R \end{aligned}$$

Clearly, the same, and correct value is obtained as a changes approaching ∞. The surface integral just balances the field squared volume integral from $r'=a$ to $r'=\infty$. Note: as in the discussion at the bottom of page 91, the formula only works when ALL the charge is enclosed in the volume so $a>R$ is required.

Griffiths 2.53

For a positive line charge (*z*-direction) of line density λ, Gauss's law gives

$$2\pi r E_r \Delta z = \frac{\lambda \Delta z}{\varepsilon_0}$$

$$E_r = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}$$

$$\phi(r) = -\int E_r dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln r = -\frac{\lambda}{4\pi\varepsilon_0}\ln r^2$$

where r is the distance from the line charge. By superposition, he total potential for the two line charges is

$$\phi_{tot} = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}.$$

Curves of constant potential have constant argument of the logarithm. Thus

$$(x+a)^2 + y^2 = K\left[(x-a)^2+y^2\right] \quad K > 1 \text{ for positive potential values}$$

$$x^2 + a^2 + y^2 + 2ax = Kx^2 + Ka^2 + Ky^2 - 2Kax$$

$$a^2(1-K) = (K-1)\left(x^2+y^2\right) - 2ax(K+1)$$

$$\left(\frac{K+1}{K-1}\right)^2 a^2 - a^2 = \left(x - a\frac{K+1}{K-1}\right)^2 + y^2 = \frac{4Ka^2}{(K-1)^2}$$

Because $K = \exp(4\pi\varepsilon_0\phi/\lambda)$, the radius and position of the center of each equipotential circle are

$$R = a\frac{2e^{2\pi\varepsilon_0\phi/\lambda}}{e^{4\pi\varepsilon_0\phi/\lambda}-1} = \frac{a}{\sinh(2\pi\varepsilon_0\phi/\lambda)}$$

$$center = a\frac{e^{4\pi\varepsilon_0\phi/\lambda}+1}{e^{4\pi\varepsilon_0\phi/\lambda}-1} = a\frac{e^{2\pi\varepsilon_0\phi/\lambda}+e^{-2\pi\varepsilon_0\phi/\lambda}}{e^{2\pi\varepsilon_0\phi/\lambda}-e^{-2\pi\varepsilon_0\phi/\lambda}} = a\frac{\cosh(2\pi\varepsilon_0\phi/\lambda)}{\sinh(2\pi\varepsilon_0\phi/\lambda)}$$

When x is negative, the argument of the logarithm goes to 1/(the argument when x is positive). Clearly, the potential reverses sign as it should.